

Regular KMS States
of
Weakly Coupled
Anharmonic Crystals
and
the Resolvent CCR Algebra

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Abstract: We consider equilibrium states of weakly coupled anharmonic quantum oscillators on \mathbf{Z} . We consider the Resolvent CCR Algebra introduced by D.Buchholtz and H.Grundling, and we show that the infinite volume limit of equilibrium states satisfies the KMS (Kubo-Martin-Schwinger) condition with regularity(= locally normal to Fock representations). Uniqueness of the KMS states is proven as well.

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1 Introduction

In this article, we consider KMS states of certain one-parameter group of automorphisms of C^* -algebra associated with canonical commutation relations (CCR), which, physically, correspond to equilibrium states of weakly coupled anharmonic quantum oscillators on \mathbf{Z} .

Let (X, σ) be a $2n$ dimensional real symplectic vector space with a non-degenerate symplectic form σ and let $\mathcal{A}_{CCR}(\sigma)$ be the CCR algebra associated with (X, σ) which is a $*$ -algebra of unbounded operators generated by formally self-adjoint elements $\Psi(f)$ ($f \in X$). $\Psi(f)$ is linear on f , satisfying CCR

$$[\Psi(f), \Psi(g)] = i\sigma(f, g)\mathbb{1}, \quad \Psi(f)^* = \Psi(f)$$

By the Heisenberg time evolution of a quantum observable Q associated with a Hamiltonian, we mean

$$\alpha_t(Q) = e^{itH}Qe^{-itH}$$

This expression is formal. Both the Hamiltonian and elements Q of $\mathcal{A}_{CCR}(\sigma)$ are unbounded operators and we have to specify domains of operators. Unless the Hamiltonian is bilinear in Boson operators $\Psi(f)$, it is likely that $\alpha_t(Q)$ is not in $\mathcal{A}_{CCR}(\sigma)$. A traditional way to handle $\mathcal{A}_{CCR}(\sigma)$ is to consider unitaries generated by $\Psi(f)$, $W(f) = e^{it\Psi(f)}$. The C^* -algebra generated by $W(f)$ is called the Weyl CCR algebra. The Weyl CCR algebra is a simple C^* -algebra and was used in study of Bose-Einstein condensation of the ideal gas. (c.f.[6])

Specialists have realized that the Weyl CCR algebra is not a C^* -algebra suitable for scattering theory and to statistical mechanics of interacting Bosons. The fundamental problem is (non-)existence of one-parameter group of automorphisms, which is physically equivalent to existence of the Heisenberg time evolution as automorphisms of a C^* -algebra. To be precise, let us consider the case of one degree of freedom. Let $X = \mathbf{R}^2$ with a symplectic basis q, p with $\sigma(q, q) = \sigma(p, p) = 0$, $\sigma(q, p) = -\sigma(p, q) = 1$. We consider the standard representation on $L^2(\mathbf{R})$, hence q is the multiplication operator and p is the differential operator. Let $v(q)$ be a potential where v is continuous, with compact support. Set $h_0 = \frac{p^2}{2}$ and $h = h_0 + v(q)$. In [8], D.Buchholz and H.Grunrdling pointed out that, for any bounded operator Q on $L^2(\mathbf{R})$,

$$e^{ith}Qe^{-ith} - e^{ith_0}Qe^{-ith_0}$$

is a compact operator. (See also [11].) This implies that any C^* -algebra on $L^2(\mathbf{R})$ invariant for both the free time evolution and the time evolution with a potential v with a compact support must contain compact operators. This rules out the Weyl CCR algebra.

In their research of mathematical foundation of the supersymmetric quantum field theory, D.Buchholz and H.Grunrdling introduced a new approach for study of the CCR algebra, *the Resolvent CCR algebra* in [7] (See [8], [9] as well) The resolvent CCR algebra is a unital C^* -algebra generated by the resolvent of the field operators

$$R(\lambda, f) = \frac{1}{i\lambda + \Psi(f)}$$

where λ is a non-zero real parameter. (See [8] for the precise definition of the resolvent CCR algebra.)

The Resolvent CCR algebra has several advantages.

- (i) Singular representations of the CCR algebra with infinite field strength can be characterized in terms of the kernel of the semi-resolvent operators.
- (ii) The Fock representation is faithful and there is a one-to-one correspondence between the regular representations of the Weyl CCR algebra and those of the resolvent CCR algebra.
- (iii) The resolvent CCR algebra for a finite quantum system contains compact operators.

Later, we will see the Hamiltonians of weakly coupled anharmonic oscillators gives rise to a one-parameter group of automorphisms on the resolvent CCR algebra and we consider KMS states.

To describe precise statements of our results, we introduce notations now. For any positive integer L we set

$$\Lambda_L = \{ j \in \mathbf{Z} \mid -L < j \leq L \} \subset \mathbf{Z}$$

$$\mathfrak{H}_{\Lambda_L} = \otimes_{k \in \Lambda_L} L^2(\mathbf{R}, dx_k)$$

and let \mathcal{B}_n be the unital abelian C^* -algebra on \mathfrak{H}_{Λ_L} generated by the unit and multiplication operators associated with the functions of the form

$$f(s_{-L+1}x_{-L+1} + s_{-L+2}x_{-L+2} \cdots + s_L x_L)$$

where $f(x)$ is a continuous function (with a single real variable) vanishing at infinity and s_{-L+1}, \dots, s_L are real constants. Let \mathfrak{R}_L be the unital C^* -algebra on \mathfrak{H}_{Λ_L} generated by the unit and operators

$$g(s_{-L+1}x_{-L+1} + \cdots + s_L x_L + t_{-L+1}p_{-L+1} + \cdots + t_L p_L)$$

where $g(x)$ is a continuous function (with a single variable) vanishing at infinity, $s_k, t_l (k, l \in \Lambda_L)$ are real constants and p_k be the quantum mechanical momentum operator $p_k = -i \frac{\partial}{\partial x_k}$. Due to the tensor product structure of \mathfrak{H}_{Λ_L} we obtain the natural inclusion $\mathfrak{R}_L \subset \mathfrak{R}_M$ if $L < M$. The inductive limit C^* -algebra of $\cup_L \mathfrak{R}_{\Lambda_L}$ is denoted by \mathfrak{R} . We call \mathfrak{R}_L and \mathfrak{R} the resolvent CCR algebra.

Theorem 1.1 *Let H_L be the Schrödinger operator defined by*

$$H_L = \sum_{k=-L+1}^L \{ p_k^2 + \omega^2 x_k^2 + V(x_k) \} + \sum_{k=-L+1}^{L-1} \varphi(x_k - x_{k+1}) \quad (1.1)$$

where the potential V and φ are rapidly decreasing smooth functions.

$\tilde{\alpha}_t^L$ defined on the Fock representation via the following equation

$$\tilde{\alpha}_t^L(Q) = e^{itH_L} Q e^{-itH_L}, \quad Q \in \mathfrak{R}_L \quad (1.2)$$

gives rise to a one-parameter group of automorphisms on \mathfrak{R}_L .

Combined with Lieb-Robinson bound techniques on Fock spaces, (c.f. [12], [14], [16], [17], [18], [21]) a C^* -dynamical systems can be introduced for weakly coupled anharmonic oscillators on the infinite lattice \mathbf{Z} .

Theorem 1.2 *The infinite volume limit*

$$\alpha_t(Q) = \lim_{L \rightarrow \infty} \tilde{\alpha}_t^L(Q) \quad (1.3)$$

exists in the norm topology of \mathfrak{R} . Let H^{free} be the Hamiltonian of decoupled oscillators

$$H^{\text{free}} = \sum_{k=-\infty}^{\infty} \{ p_k^2 + \omega^2 x_k^2 + V(x_k) \} \quad (1.4)$$

and set $\alpha_t^{\text{free}}(Q) = e^{itH^{\text{free}}} Q e^{-itH^{\text{free}}}$.

Then, $\alpha_t \circ \alpha_{-t}^{\text{free}}(Q)$ and $\alpha_{-t}^{\text{free}} \circ \alpha_t(Q)$ are continuous in t for the norm topology of \mathfrak{R} .

Independently, D.Buchholz obtained the infinite volume limit of dynamics for more general models by different methods in [10]. We believe that application of Lieb-Robinson bound techniques in itself will be useful for more advanced research in future.

The main results of this paper is uniqueness of the regular KMS states of weakly coupled quantum oscillators. Uniqueness of the KMS states for one-dimensional quantum systems is a well-known fact, however, in our case, the time evolution α_t is not norm continuous in t . α_t is continuous in weak topology on the GNS representations which is locally quasi-equivalent to the standard Fock representation. We will see that there exists a infinite volume limit ψ of KMS states of finite systems such that the restriction ψ to each finite volume is normal to the Fock representation and $\psi(Q\alpha_t(R))$ is continuous in t . By regularity of states we mean states locally normal to the standard Fock state. (See Definition 5.1 and Theorem 5.8 below.)

Theorem 1.3 *The regular KMS state associated with the Hamiltonian (1.1) exists, and is unique.*

Statistical Mechanics of anharmonic crystals with the Hamiltonian (1.5) defined below has been extensively studied by several people. (c.f. [1], [15] and the references therein)

$$H = \sum_{j \in \mathbf{Z}^d} \{ p_j^2 + V(x_j) \} + \sum_{j, i \in \mathbf{Z}^d, ||i-j||=1} |x_i - x_j|^2 \quad (1.5)$$

where V is a polynomial giving rise to a double well potential. Note that, in our anharmonic crystal, Bose particles are fixed on the lattice sites and they are distinguishable.

Results obtained so far are based on perturbation theory and for developing a general theory a missing point is a suitable C^* -algebra describing full quantum observables. We believe that the resolvent CCR algebra introduced by

D.Buchholz and H.Grunrdling is the right staff for handling the full quantum system including momentum operators. We hope the results of this article is the first step of understanding equilibrium states of anharmonic crystals.

2 Resolvent CCR algebra

In this section, we introduce the notation and recall the results of the resolvent CCR algebra in [8].

For a given subset $\Lambda \subset \mathbf{Z}$, we denote $c_c(\Lambda)$ by the space of all finitely supported function $f : \Lambda \rightarrow \mathbf{C}$. We define the symplectic form σ on $c_c(\Lambda)$ by $\sigma(f, g) = \text{Im} \langle f, g \rangle_{\ell^2}$ for $f, g \in c_c(\Lambda)$, where $\langle \cdot, \cdot \rangle_{\ell^2}$ is the canonical inner product on $\ell^2(\mathbf{Z})$. Then $c_c(\Lambda)$ equipped with σ is also a symplectic space.

We consider the Hilbert space \mathfrak{H}_Λ associated with any finite subset Λ of \mathbf{Z} defined by

$$\mathfrak{H}_\Lambda = \bigotimes_{k \in \Lambda} L^2(\mathbf{R}, dx_k),$$

where dx_k is the Lebesgue measure on \mathbf{R} . To simplify the notation, for any finite subsets $\Lambda \subset \Gamma \subset \mathbf{Z}$, we identify the linear operator A on \mathfrak{H}_Λ with the linear operator $A \otimes \mathbb{1}_{\Gamma \setminus \Lambda}$ on \mathfrak{H}_Γ , where $\mathbb{1}_{\Gamma \setminus \Lambda}$ is the identity operator on $\mathfrak{H}_{\Gamma \setminus \Lambda}$. Thus, for any finite subset $\Lambda \subset \mathbf{Z}$, we identify the multiplication operator x_k on $L^2(\mathbf{R}, dx_k)$, $k \in \Lambda$, and $x_k \otimes \mathbb{1}_{\Lambda \setminus \{k\}}$ on \mathfrak{H}_{Λ_L} . Also, we identify the differential operator $p_k = -i \frac{\partial}{\partial x_k}$ and $p_k \otimes \mathbb{1}_{\Lambda \setminus \{k\}}$. We denote the trace on \mathfrak{H}_L by Tr_L .

For any subset Λ of \mathbf{Z} , we denote $\mathcal{W}(\Lambda)$ and $\mathfrak{R}(\Lambda)$ by the Weyl CCR algebra and the resolvent CCR algebra over $(c_c(\Lambda), \sigma)$, respectively. The definitions of the Weyl CCR algebra and the resolvent CCR algebra are as follows.

The Weyl CCR algebra is the C^* -algebra generated by $W(f)$, $f \in c_c(\Lambda)$, satisfying

$$\begin{aligned} W(f)^* &= W(-f), \\ W(f)W(g) &= e^{-i \frac{\sigma(f, g)}{2}} W(f + g) \end{aligned}$$

for all $f, g \in c_c(\Lambda)$ (see e.g. [6, Theorem 5.2.8.]).

The resolvent CCR algebra $\mathfrak{R}(\Lambda)$ is the universal C^* -algebra generated by $R(\lambda, f)$, $\lambda \in \mathbf{R} \setminus \{0\}$, $f \in c_c(\Lambda)$, satisfying

$$R(\lambda, 0) = -\frac{i}{\lambda}, \quad (2.1)$$

$$R(\lambda, f)^* = R(-\lambda, f), \quad (2.2)$$

$$\nu R(\nu \lambda, \nu f) = R(\lambda, f), \quad (2.3)$$

$$R(\lambda, f) - R(\mu, f) = i(\mu - \lambda)R(\lambda, f)R(\mu, f), \quad (2.4)$$

$$[R(\lambda, f), R(\mu, g)] = i\sigma(f, g)R(\lambda, f)R(\mu, g)^2R(\lambda, f), \quad (2.5)$$

$$\begin{aligned} R(\lambda, f)R(\mu, g) &= R(\lambda + \mu, f + g)\{R(\lambda, f) + R(\mu, g) \\ &\quad + i\sigma(f, g)R(\lambda, f)R(\mu, g)\} \quad (\lambda \neq -\mu), \end{aligned} \quad (2.6)$$

where $\lambda, \mu, \nu \in \mathbf{R} \setminus \{0\}$ and $f, g \in c_c(\Lambda)$. (See [8].) For any subset $\Lambda \subset \mathbf{Z}$, the resolvent CCR algebra $\mathfrak{R}(\Lambda)$ is the inductive limit of the net of all finite dimensional non-degenerate symplectic subspaces of $c_c(\Lambda)$ [8, Theorem 4.9 (ii)].

For any positive integer L , we set $\Lambda_L = \{j \in \mathbf{Z} \mid -L < j \leq L\}$. For simplicity, we set $\mathfrak{R} = \mathfrak{R}(\mathbf{Z})$ and $\mathfrak{R}_L = \mathfrak{R}(\Lambda_L)$. Also, we set $\mathfrak{R}_{L^c} = \mathfrak{R}(\mathbf{Z} \setminus \Lambda_L)$ and $\mathfrak{R}_{L' \setminus L} = \mathfrak{R}(\Lambda_{L'} \setminus \Lambda_L)$, for any positive integers $L \leq L'$. For the Weyl CCR algebra, we also set $\mathcal{W} = \mathcal{W}(\mathbf{Z})$, $\mathcal{W}_L = \mathcal{W}(\Lambda_L)$, $\mathcal{W}_{L^c} = \mathcal{W}(\mathbf{Z} \setminus \Lambda_L)$ and $\mathcal{W}_{L' \setminus L} = \mathcal{W}(\Lambda_{L'} \setminus \Lambda_L)$, for any positive integers $L \leq L'$.

Let π_0 be the Schrödinger representation of $\mathfrak{R}(\Lambda)$ on \mathfrak{H}_Λ . Due to [8, Theorem 4.10], π_0 is a faithful representation of $\mathfrak{R}(\Lambda)$.

3 Lieb-Robinson bounds and limiting dynamics for the resolvent CCR algebra

In this section, we prove the Lieb-Robinson bounds for weakly coupled anharmonic quantum oscillators on the resolvent CCR algebra. First, we introduce the notation.

For any $L \in \mathbf{N}$ and positive constant $\omega \geq 0$, let H_L^h be the self-adjoint operator on \mathfrak{H}_{Λ_L} defined by

$$H_L^h = \sum_{k \in \Lambda_L} (p_k^2 + \omega^2 x_k^2).$$

We define the automorphism $\tilde{\alpha}_t^{h,L}$ on $\mathcal{B}(\mathfrak{H}_{\Lambda_L})$ by

$$\tilde{\alpha}_t^{h,L}(Q) = e^{itH_L^h} Q e^{-itH_L^h}, \quad Q \in \mathcal{B}(\mathfrak{H}_{\Lambda_L}).$$

Since the automorphism $\tilde{\alpha}_t^{h,L}$ induce the symplectic transform on $(c_c(\Lambda_L), \sigma)$, $\tilde{\alpha}_t^{h,L}$ is an automorphism on $\pi_0(\mathfrak{R}_L)$. Let Φ be the map from any finite subset Λ of \mathbf{Z} to $\mathcal{B}(\mathfrak{H}_\Lambda)$ defined by

$$\Phi(\Lambda) = \begin{cases} V(x_k) & (\Lambda = \{k\}) \\ \varphi(x_k - x_{k+1}) & (\Lambda = \{k, k+1\}) \\ 0 & (\text{otherwise}) \end{cases}, \quad (3.1)$$

where V and φ are real valued Schwarz functions on \mathbf{R} . The function V represent anharmonicity of the potential of the system and φ corresponds to the nearest neighbor interaction of particles. For any finite subset $\Gamma \subset \mathbf{Z}$, we set $\Upsilon(\Gamma) = \sum_{\Lambda \subset \Gamma} \Phi(\Lambda)$. For simplicity, we set $\Upsilon_L = \Upsilon(\Lambda_L)$, $L \in \mathbf{N}$ and $\Upsilon_{L' \setminus L} = \Upsilon(\Lambda_{L'} \setminus \Lambda_L)$ whenever $L \leq L'$. Let H_L be the self-adjoint operator on \mathfrak{H}_{Λ_L} defined by

$$H_L = H_L^h + \Upsilon_L = \sum_{k \in \Lambda_L} (p_k^2 + \omega^2 x_k^2 + V(x_k)) + \sum_{k, k+1 \in \Lambda_L} \varphi(x_k - x_{k+1}). \quad (3.2)$$

3.1 Proof of Theorem 1.1

As the Fock representation is faithful, we consider the convergence in strong and norm topologies of operator valued integral on the Fock space. Let $U_L(t)$ be the unitary operator on \mathfrak{H}_{Λ_L} defined by $U_L(t) = e^{itH_L} e^{-itH_L^h}$. By using the Dyson series expansion of $U_L(t)$, we obtain

$$U_L(t) = \mathbb{1} + \sum_{n \geq 1} i^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \tilde{\alpha}_{t_n}^{h,L}(\Upsilon_L) \cdots \tilde{\alpha}_{t_1}^{h,L}(\Upsilon_L).$$

(See e.g. [6, Theorem 3.1.33].) Generally speaking, the above integral makes sense in the weak topology on the Fock space. However, in our current situation, D.Buchholz and H.Grundling have shown that

$$\int_0^t \tilde{\alpha}_{t_n}^{h,L}(V(x_k)), \quad \int_0^t \tilde{\alpha}_{t_n}^{h,L}(\varphi(x_k - x_{k-1}))$$

are norm continuous families of elements of the resolvent CCR algebra. (See [8, Proposition 6.1] and [8, Proof of Proposition 7.1]) and as a consequence, the Dyson series converge in the norm topology.

Thus, $\tilde{\alpha}_t^L$ is a one-parameter group of automorphism on $\pi_0(\mathfrak{R}_L)$. ■

3.2 Lieb-Robinson bounds and limiting dynamics

Next, we consider the Lieb-Robinson bound of the automorphism $\tilde{\alpha}_t^L$. Before we prove the Lieb-Robinson bound of $\tilde{\alpha}_t^L$, we introduce the following notations. (See also [19], [20] and [21].)

We set

$$H_L^{\text{free}} = \sum_{k \in \Lambda_L} (p_k^2 + \omega^2 x_k^2 + V(x_k))$$

and

$$\alpha_t^{\text{free},L}(Q) = e^{itH_L^{\text{free}}} Q e^{-itH_L^{\text{free}}}, \quad Q \in \mathfrak{R}_L.$$

For any subset $\Gamma \subset \mathbf{Z}$ and any finite subset $\Lambda \subset \Gamma$, we set

$$S_\Gamma(\Lambda) = \{X \subset \Gamma \mid X \cap \Lambda \neq \emptyset, X \cap (\Gamma \setminus \Lambda) \neq \emptyset\}.$$

If $\Gamma = \mathbf{Z}$, then we denote $S(\Lambda)$ by $S_{\mathbf{Z}}(\Lambda)$. Let $\partial_\Phi \Lambda$ be the subset of Λ defined by

$$\partial_\Phi \Lambda = \{x \in \Lambda \mid \text{for some } X \in S(\Lambda) \text{ with } x \in X, \Phi(X) \neq 0\}.$$

For any finite subsets $\Gamma_1, \Gamma_2 \subset \mathbf{Z}$, we set

$$D(\Gamma_1, \Gamma_2) = \min \left\{ \sum_{x \in \partial_\Phi \Gamma_1} \sum_{y \in \Gamma_2} \frac{1}{1 + |x - y|}, \sum_{x \in \Gamma_1} \sum_{y \in \partial_\Phi \Gamma_2} \frac{1}{1 + |x - y|} \right\}.$$

For $L \in \mathbf{N}$, we set $C = 4 \sum_{x \in \mathbf{Z}} \frac{1}{(1+|x|)^2}$. We define the norm $\|\cdot\|_{\text{int}}$ of the map Φ by

$$\|\Phi\|_{\text{int}} = \sup_{\substack{x, y \in \mathbf{Z} \\ x \neq y}} \frac{1}{(1+|x-y|)^2} \sum_{\substack{\Lambda: x, y \in \Lambda \subset \mathbf{Z} \\ |\Lambda| < \infty}} \|\Phi(\Lambda)\|$$

where $|\Lambda|$ is the number of elements of Λ .

Lemma 3.1 *Let Γ_1 and Γ_2 be finite disjoint subsets of \mathbf{Z} . For any finite subset Λ_L of \mathbf{Z} , $L \in \mathbf{N}$, with $\Gamma_1 \cup \Gamma_2 \subset \Lambda_L$ and arbitrary $Q \in \pi_0(\mathfrak{R}(\Gamma_1))$ and $R \in \pi_0(\mathfrak{R}(\Gamma_2))$, it follows that*

$$\left\| \left[\tilde{\alpha}_t^L(\tilde{\alpha}_{-t}^{\text{free}, L}(Q)), R \right] \right\| \leq \frac{2\|Q\|\|R\|}{C} \left(e^{2\|\Phi\|_{\text{int}} C|t|} - 1 \right) D(\Gamma_1, \Gamma_2). \quad (3.3)$$

holds for any $t \in \mathbf{R}$.

Proof. Let $\varphi(\Lambda_L) = \sum_{k, k+1 \in \Lambda_L} \varphi(x_{k+1} - x_k)$ and for any finite subset $\Lambda \subset \mathbf{Z}$ let $\mathfrak{Q}(\Lambda)$ be the norm dense subset of $\pi_0(\mathfrak{R}(\Lambda))$ defined by

$$\mathfrak{Q}(\Lambda) = \text{span}\{\pi_0(R(\lambda_1, f_1) \cdots R(\lambda_n, f_n)) \mid \lambda_i \in \mathbf{R} \setminus \{0\}, f_i \in c_c(\Lambda), i = 1, \dots, n, n \in \mathbf{N}\}.$$

Put

$$F(t) = \left[\tilde{\alpha}_t^L \left(\tilde{\alpha}_{-t}^{\text{free}, L}(Q) \right), R \right], \quad (3.4)$$

for $Q \in \mathfrak{Q}(\Gamma_1)$ and $R \in \mathfrak{Q}(\Gamma_2)$. Since $R(\lambda, f)$ preserves the domain of multiplication and differential operators by [8, Theorem 4.2 (i)] and the Schwarz function in \mathfrak{H}_{Λ_L} is analytic elements for the operator $\sum_{k \in \Lambda_L} (p_k^2 + \omega^2 x_k^2 + V(x_k))$, the function F is strongly differentiable. Thus, the derivation of F is

$$\frac{d}{dt} F(t) = i [\tilde{\alpha}_t^L(\varphi(\Lambda_L)), F(t)] - i \left[\tilde{\alpha}_t^L \left(\tilde{\alpha}_{-t}^{\text{free}, L}(Q) \right), [\tilde{\alpha}_t^L(\varphi(\Lambda_L)), R] \right]. \quad (3.5)$$

Since $\tilde{\alpha}_t^L$ is strongly continuous on \mathfrak{H}_{Λ_L} , the solution $F(t)$ of the equation (3.5) satisfies the following estimate by [21, Lemma 2.2]:

$$\|F(t)\| \leq \|F(0)\| + \int_0^{|t|} ds \left\| \left[\tilde{\alpha}_s^L \left(\tilde{\alpha}_{-s}^{\text{free}, L}(Q) \right), [\tilde{\alpha}_s^L(\varphi(\Lambda_L)), R] \right] \right\|.$$

Since for any subset $\Lambda \subset \mathbf{Z}$, $\mathfrak{Q}(\Lambda)$ is a norm dense subset in $\pi_0(\mathfrak{R}(\Lambda))$, the above inequality holds for any elements of $Q \in \pi_0(\mathfrak{R}(\Gamma_1))$ and $R \in \pi_0(\mathfrak{R}(\Gamma_2))$.

By the proof of [21, Theorem 3.1], we get

$$\left\| \left[\tilde{\alpha}_t^L(\tilde{\alpha}_{-t}^{\text{free}, L}(Q)), R \right] \right\| \leq \frac{2\|Q\|\|R\|}{C} \left(e^{2\|\Phi\|_{\text{int}} C|t|} - 1 \right) D(X, Y). \quad (3.6)$$

for any $Q \in \pi_0(\mathfrak{R}(\Gamma_1))$ and $R \in \pi_0(\mathfrak{R}(\Gamma_2))$. ■

We define the automorphisms α_t^L and $\alpha_t^{h, L}$ ($t \in \mathbf{R}$) on \mathfrak{R}_L by

$$\alpha_t^L(Q) = \pi_0^{-1}(e^{itH_L})Q\pi_0^{-1}(e^{-itH_L}), \quad (3.7)$$

$$\alpha_t^{\text{free}, L}(Q) = \pi_0^{-1}(e^{itH_L^{\text{free}}})Q\pi_0^{-1}(e^{-itH_L^{\text{free}}}) \quad (3.8)$$

for $Q \in \mathfrak{R}_L$. Note that the Schrödinger representation π_0 is faithful representation. Thus, we get the following.

Corollary 3.2 *Let Γ_1 and Γ_2 be finite disjoint subsets of \mathbf{Z} . For any finite Λ_L with $\Gamma_1 \cup \Gamma_2 \subset \Lambda_L$ and arbitrary $Q \in \mathfrak{R}(\Gamma_1)$ and $R \in \mathfrak{R}(\Gamma_1)$, it follows that*

$$\left\| \left[\alpha_t^L(\alpha_{-t}^{\text{free}, L}(Q)), R \right] \right\| \leq \frac{2 \|Q\| \|R\|}{C} \left(e^{2\|\Phi\|_{\text{int}} C |t|} - 1 \right) D(\Gamma_1, \Gamma_2). \quad (3.9)$$

holds for all $t \in \mathbf{R}$.

Theorem 3.3 *For any $t \in \mathbf{R}$, $L \in \mathbf{N}$ and $Q \in \mathfrak{R}_L$, the norm limit*

$$\lim_{N \rightarrow \infty} \alpha_t^N(Q) = \alpha_t(Q) \quad (3.10)$$

exists and the convergence is uniform for t in compact sets.

Proof. The assertion follows from the proof of [21, Theorem 4.1] and the above corollary. ■

Finally, we give the proof of Theorem 1.2.

3.3 Proof of Theorem 1.2.

The existence of the infinite volume limit is proven in Theorem 3.3. Thus, we prove the continuity of $\alpha_t \circ \alpha_{-t}^{\text{free}}$ in $t \in \mathbf{R}$. We may assume that $t \in [0, T]$. By the Dyson series expansions of $e^{itH_L} e^{-itH_L^{\text{free}}}$, we obtain

$$\begin{aligned} \left\| e^{itH_L} e^{-itH_L^{\text{free}}} - \mathbb{1} \right\| &= \left\| \sum_{n \geq 1} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \alpha_{t_n}(\varphi(\Lambda_L)) \cdots \alpha_{t_1}(\varphi(\Lambda_L)) \right\| \\ &\leq e^{2L\|\varphi\|_\infty T} - 1, \end{aligned} \quad (3.11)$$

where $\varphi(\Lambda_L) = \sum_{k, k+1 \in \Lambda_L} \varphi(x_{k+1} - x_k)$. Note that $\|\Phi\|_{\text{int}} \leq \frac{1}{4} \|\varphi\|_\infty$, where $\|\varphi\|_\infty$ is the supremum norm of φ . By using the estimate (83) of the proof of [21, Theorem 4.1], for any $L \in \mathbf{N}$, $L \leq N \leq N'$ and $Q \in \mathfrak{R}_L$, we have

$$\begin{aligned} &\left\| \alpha_t^{N'} \circ \alpha_t^{\text{free}, N'}(Q) - \alpha_t^N \circ \alpha_t^{\text{free}, N}(Q) \right\| \\ &\leq \frac{1}{2} T (1 + e^{\frac{1}{2} C \|\varphi\|_\infty T}) \|Q\| \sum_{k \in \Lambda_L} \sum_{l \in \Lambda_{N'} \setminus \Lambda_N} \frac{1}{(1 + |k - l|)^2}. \end{aligned}$$

Thus, when $N = L$ and $N' \rightarrow \infty$, we obtain

$$\left\| \alpha_t \circ \alpha_t^{\text{free}}(Q) - \alpha_t^L \circ \alpha_t^{\text{free}, L}(Q) \right\| \leq T (1 + e^{\frac{1}{2} C \|\varphi\|_\infty T}) \|Q\| LC. \quad (3.12)$$

By (3.11) and (3.12), it follows that

$$\begin{aligned} \left\| \alpha_t \circ \alpha_t^{\text{free}}(Q) - Q \right\| &\leq \left\| \alpha_t \circ \alpha_t^{\text{free}}(Q) - \alpha_t^L \circ \alpha_t^{\text{free}, L}(Q) \right\| \\ &\quad + 2 \left\| e^{itH_L} e^{-itH_L^{\text{free}}} - \mathbb{1} \right\| \|Q\| \\ &\leq T (1 + e^{\frac{1}{2} C \|\varphi\|_\infty T}) \|Q\| LC + 2(e^{2TL\|\varphi\|_\infty} - 1) \|Q\|. \end{aligned}$$

Thus, we are done. ■

4 Regular states of the resolvent CCR algebra

In this section, we consider regular states on \mathfrak{R}_L , $L \in \mathbf{N}$, or \mathfrak{R} . Recall that a state ψ on \mathfrak{R}_L or \mathfrak{R} is regular, if and only if $\ker(\pi_\psi(R(\lambda, f))) = \{0\}$ for any $\lambda \in \mathbf{R} \setminus \{0\}$ and $f \in c_c(\Lambda_L)$ or $f \in c_c(\mathbf{Z})$, respectively, where π_ψ is the GNS representation associated with ψ . (c.f. [8, Definition 4.3]) In another word , $\pi_\psi(R(\lambda, f))$ is the resolvent of a closed operator if ψ is regular. Note that there is a one-to-one correspondence between a regular state of \mathfrak{R} and that of \mathcal{W} . (See [8, Corollary 4.4.] .) and by abuse of notations, we employ the same notation , ψ or φ etc. for the regular states of \mathfrak{R} and \mathcal{W} .

The following claims are straight forward implication of the Stone-von Neumann uniqueness theorem. (See e.g. [6, Corollary 5.2.15].)

Lemma 4.1 *Let ψ be a regular state of \mathfrak{R}_L . Then. ψ is normal with respect to the Fock representation.*

Corollary 4.2 *Let ψ be a regular state on \mathfrak{R}_L . Then there exists a positive trace class operator ρ on \mathfrak{H}_{Λ_L} such that $\text{Tr}_L(\rho) = 1$ and $\psi(Q) = \text{Tr}_L(\rho\pi_0(Q))$, where π_0 is the Schrödinger representation of \mathfrak{R}_L and Tr_L is the trace on \mathfrak{H}_{Λ_L} .*

Lemma 4.3 *Let ψ be a regular state on \mathfrak{R}_L , $L \in \mathbf{N}$. Let $(\mathfrak{H}_\psi, \pi_\psi, \xi_\psi)$ be the GNS representation of ψ . Put $\mathfrak{K}(\Lambda_L) = \pi_0^{-1}(\mathcal{K}(\mathfrak{H}_{\Lambda_L}))$, where $\mathcal{K}(\mathfrak{H})$ is the set of all compact operator on a Hilbert space \mathfrak{H} . Then, $\pi_\psi(\mathfrak{K}(\Lambda_L))$ is weakly dense in $\pi_\psi(\mathfrak{R}_L)''$.*

Proposition 4.4 *Let ψ be a regular state on \mathfrak{R}_L , $L \in \mathbf{N}$. Then, $\psi(\alpha_t^L(Q)R)$ is continuous on $t \in \mathbf{R}$ for any $Q, R \in \mathfrak{R}_L$ where α_t^L is defined in (3.7).*

Proof. In $\mathcal{B}(\mathfrak{H}_{\Lambda_L})$, we consider the Dyson series of $U_L(t) = e^{itH_L}e^{-itH_L^h}$ and $U_L(t) - \mathbb{1}$ has the following estimate:

$$\begin{aligned} \|U_L(t) - \mathbb{1}\| &= \left\| \sum_{n \geq 1} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \tilde{\alpha}_{t_n}^{h,L}(\Upsilon_L) \cdots \tilde{\alpha}_{t_1}^{h,L}(\Upsilon_L) \right\| \\ &\leq (e^{|t|\|\Upsilon_L\|} - 1). \end{aligned}$$

Note that $\tilde{\alpha}_t^{h,L}$ preserve the set of all of compact operators $\mathcal{K}(\mathfrak{H}_{\Lambda_L})$ and for $Q \in \mathcal{K}(\mathfrak{H}_{\Lambda_L})$, $\tilde{\alpha}_t^{h,L}(Q)$ is norm continuous. Since for $Q \in \mathcal{K}(\mathfrak{H}_{\Lambda_L})$, we obtain

$$\begin{aligned} \|e^{itH_L}Qe^{-itH_L} - Q\| &= \left\| U_L(t)e^{itH_L^h}Qe^{-itH_L^h}U_L(t)^{-1} - Q \right\| \\ &\leq 2(e^{|t|\|\Upsilon_L\|} - 1)\|Q\| + \left\| \tilde{\alpha}_t^{h,L}(Q) - Q \right\|. \quad (4.1) \end{aligned}$$

Since the Schrödinger representation π_0 is faithful, for any $Q \in \mathcal{K}(\mathfrak{H}_{\Lambda_L})$, $\tilde{\alpha}_t^L(Q)$ is norm continuous for $t \in \mathbf{R}$. By Corollary 4.2 and Lemma 4.3, $\psi(R\alpha_t^L(Q))$ is continuous for $R, Q \in \mathfrak{R}_L$. ■

Next let us recall the definition of quasi-containment. Let \mathcal{A} be a C^* -algebra and let (\mathfrak{H}_1, π_1) and (\mathfrak{H}_2, π_2) be nondegenerate representations of \mathcal{A} . The representations π_1 and π_2 is quasi-equivalent, if there exists an isomorphism $\gamma : \pi_1(\mathcal{A})'' \mapsto \pi_2(\mathcal{A})''$ such that $\gamma(\pi_1(A)) = \pi_2(A)$ for all $A \in \mathcal{A}$ (see also [6, Definition 2.4.25] and [6, Theorem 2.4.26]). If a subrepresentation of π_1 is quasi-equivalent to π_2 , then π_1 is quasi-contain π_2 . The next lemma is essentially due to [2, Lemma 1].

Lemma 4.5 *Let ψ_1 and ψ_2 be regular states on \mathfrak{R} and $(\mathfrak{H}_1, \pi_1, \xi_1)$ and $(\mathfrak{H}_2, \pi_2, \xi_2)$ be the GNS representations associated with ψ_1 and ψ_2 , respectively. If π_1 does not quasi-contain π_2 , then there exists a sequence of projections $e_m \in \bigcup_{L \in \mathbb{N}} \mathfrak{R}_L$ such that*

$$\lim_m \psi_1(e_m) = 0, \quad (4.2)$$

$$\lim_m \psi_2(e_m) = a > 0. \quad (4.3)$$

Proof. Put $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$, $\pi = \pi_1 \oplus \pi_2$, $\tilde{\xi}_1 = \xi_1 \oplus 0$, $\tilde{\xi}_2 = 0 \oplus \xi_2$ and $\mathfrak{M} = \pi(\mathfrak{R})''$. Note that π is a regular representation of \mathfrak{R} . Let E_1 and E_2 be the projections from \mathfrak{H} onto \mathfrak{H}_1 and \mathfrak{H}_2 , respectively. By the assumption, there exists a central projection $E \in \mathfrak{M} \cap \mathfrak{M}'$ such that $EE_1 = 0$, $E \leq E_2$. By Kaplansky's density theorem, there exists self-adjoint elements $a_n \in \mathfrak{R}_{L(n)}$ such that

$$\text{st-}\lim_n \pi(a_n) = E,$$

where the st-lim is the strong limit in \mathfrak{H} . By the regularity of π , Lemma 4.3 and Kaplansky's density theorem, there exists self-adjoint elements $b_m^{(n)} \in \mathfrak{R}(\Lambda_{L(n)})$ such that

$$\text{st-}\lim_m \pi(b_m^{(n)}) = \pi(a_n).$$

Note that the spectral projections of $b_m^{(n)}$ are also contained in \mathfrak{R}_L . Let e_n be the spectral projection of $b_m^{(n)}$ for an interval $[1 - \delta, 1 + \delta]$, where $\delta \in (0, 1)$ is fixed. Then, $e_n \in \mathfrak{R}(\Lambda_{L(n)}) \subset \mathfrak{R}_{L(n)}$ and $\text{st-}\lim_n \pi(e_n) = E$.

Thus, $\lim_n \psi_1(e_n) = 0$, $\lim_n \psi_2(e_n) = a > 0$. ■

4.1 Relative entropy

In this subsection, we recall the definition and the properties of the relative entropy of normal states of von Neumann algebras.

The relative entropy for positive normal linear functionals on a von Neumann algebra was introduced by H. Araki in [3] and [4]. Let ψ_1 and ψ_2 be positive normal linear functionals over a von Neumann algebra \mathfrak{M} . Due to the theory of standard form of von Neumann algebra, there exists a Hilbert space \mathfrak{H} and $\xi_1, \xi_2 \in \mathfrak{H}$ such that $\psi_1(a) = \langle \xi_1, a\xi_1 \rangle$ and $\psi_2(a) = \langle \xi_2, a\xi_2 \rangle$, $a \in \mathfrak{M}$.

Let S_{ξ_2, ξ_1} be the closable densely defined conjugate linear operator S_{ξ_2, ξ_1} defined by

$$S_{\xi_2, \xi_1} a \xi_1 = a^* \xi_2,$$

for $a \in \mathfrak{M}$. The relative modular operator Δ_{ξ_2, ξ_1} is, by definition,

$$\Delta_{\xi_2, \xi_1} = S_{\xi_2, \xi_1}^* \overline{S_{\xi_2, \xi_1}}, \quad (4.4)$$

where $\overline{S_{\xi_2, \xi_1}}$ is the closure of the operator S_{ξ_2, ξ_1} . We denote the projection onto $\overline{\mathfrak{M}'\xi_1}$ and $\overline{\mathfrak{M}'\xi_2}$ on \mathfrak{H} by $s(\psi_1)$ and $s(\psi_2)$, respectively. We define the relative entropy $S_A(\psi_1, \psi_2)$ of ψ_1 and ψ_2 by

$$S_A(\psi_1, \psi_2) = \begin{cases} -\int_0^\infty \log(\lambda) d(\xi_1, E(\lambda)\xi_1), & \text{if } (s(\psi_1) \leq s(\psi_2)) \\ \infty & \text{otherwise} \end{cases}. \quad (4.5)$$

A. Uhlmann introduced the relative entropy for positive linear functionals on a (not necessarily normed) $*$ -algebra. (c.f. [24].) We denote the relative entropy defined by A. Uhlmann by S_U . The definition of relative entropy of A. Uhlmann and that of H. Araki coincide for any positive normal linear functionals on any von Neumann algebra (see [13].). We recall that basic properties of relative entropy.

Lemma 4.6 ([13, Lemma 3.1]) *Let \mathcal{A} be a unital C^* -algebra and π be a non-degenerate representation of \mathcal{A} on a Hilbert space. If ψ_1 and ψ_2 are positive linear functionals of \mathcal{A} with normal extensions $\widehat{\psi_1}$ and $\widehat{\psi_2}$ to $\pi(\mathcal{A})''$ such that $\psi_1(A) = \widehat{\psi_1}(\pi(A))$ and $\psi_2(A) = \widehat{\psi_2}(\pi(A))$, $A \in \mathcal{A}$, then*

$$S_U(\psi_1, \psi_2) = S_A(\widehat{\psi_1}, \widehat{\psi_2}). \quad (4.6)$$

Thus, in this paper, by the relative entropy of states on a unital C^* -algebra, we mean the relative entropy of normal extension of states to the von Neumann algebra associated with the GNS representation. More precisely, let \mathcal{A} be a C^* -algebra and π be a non-degenerate representation of \mathcal{A} on a Hilbert space \mathfrak{H} . Let $\widehat{\psi_1}$ and $\widehat{\psi_2}$ be positive normal linear functionals on $\pi(\mathcal{A})''$. We set $\psi_1(A) = \widehat{\psi_1}(\pi(Q))$ and $\psi_2(A) = \widehat{\psi_2}(\pi(Q))$ for $Q \in \mathcal{A}$. The relative entropy $S(\psi_1, \psi_2)$ of ψ_1 and ψ_2 is defined by

$$S(\psi_1, \psi_2) = S_A(\widehat{\psi_1}, \widehat{\psi_2}).$$

Lemma 4.7 *Let ψ_1 and ψ_2 be a regular states on \mathfrak{R} . Then*

$$S(\psi_1 \upharpoonright_{\mathfrak{R}_L}, \psi_2 \upharpoonright_{\mathfrak{R}_L}) = S(\widehat{\psi_1} \upharpoonright_{\pi_0(\mathfrak{R}_L)''}, \widehat{\psi_2} \upharpoonright_{\pi_0(\mathfrak{R}_L)''}). \quad (4.7)$$

Proof. By Corollary 4.2, there exists trace class operators ρ_1 and ρ_2 on \mathfrak{H}_{Λ_L} . Thus, we set $\widehat{\psi_1}(\pi_0(Q)) = \text{Tr}_L(\rho_1 \pi_0(Q))$ and $\widehat{\psi_2}(\pi_0(Q)) = \text{Tr}_L(\rho_2 \pi_0(Q))$ for $Q \in \mathfrak{R}_L$. Then, we obtain

$$\begin{aligned} S(\psi_1 \upharpoonright_{\mathfrak{R}_L}, \psi_2 \upharpoonright_{\mathfrak{R}_L}) &= S(\widehat{\psi_1} \circ \pi_0 \upharpoonright_{\mathfrak{R}_L}, \widehat{\psi_2} \circ \pi_0 \upharpoonright_{\mathfrak{R}_L}) \\ &= S(\widehat{\psi_1} \upharpoonright_{\pi_0(\mathfrak{R}_L)''}, \widehat{\psi_2} \upharpoonright_{\pi_0(\mathfrak{R}_L)''}). \end{aligned}$$

■

Lemma 4.8 ([22, Corollary 5.12 (iii)]) *Let $\mathfrak{N} \subset \mathfrak{M}$ be von Neumann algebras and ψ_1 and ψ_2 be normal states on \mathfrak{M} . Assume there exists a norm one projection from \mathfrak{M} to \mathfrak{N} . Then*

$$0 \leq S(\psi_1 \upharpoonright_{\mathfrak{N}}, \psi_2 \upharpoonright_{\mathfrak{N}}) \leq S(\psi_1, \psi_2). \quad (4.8)$$

By the same argument as that in [2], we have the following.

Lemma 4.9 *Let ψ_1 and ψ_2 be regular states on \mathfrak{R} . If*

$$\sup_{L \in \mathbf{N}} S(\psi_1 \upharpoonright_{\mathfrak{R}_L}, \psi_2 \upharpoonright_{\mathfrak{R}_L}) \equiv \mu < \infty, \quad (4.9)$$

then π_2 quasi contains π_1 where π_j is the GNS representation of \mathfrak{R} associated with ψ_j , $j = 1, 2$.

Proof. Assume that π_2 does not quasi-contain π_1 . By Lemma 4.5, there exists a sequence of projections $e_n \in \mathfrak{R}_{L(n)}$ such that

$$\begin{aligned} \lim_n \psi_1(e_n) &= a > 0, \\ \lim_n \psi_2(e_n) &= 0. \end{aligned}$$

Then,

$$-\psi_1(e_n) \log \psi_2(e_n) \rightarrow \infty.$$

Consider the C^* -subalgebra \mathcal{B}_n of $\mathfrak{R}_{L(n)}$ generated by e_n and $1 - e_n$. By Lemma 4.7 and Lemma 4.8,

$$\begin{aligned} S(\psi_1 \upharpoonright_{\mathfrak{R}_L}, \psi_2 \upharpoonright_{\mathfrak{R}_L}) &= S(\widehat{\psi_1 \upharpoonright_{\pi_0(\mathfrak{R}_{L(n))''}}, \widehat{\psi_2 \upharpoonright_{\pi_0(\mathfrak{R}_{L(n))''}}} \\ &\geq S(\widehat{\psi_1 \upharpoonright_{\pi_0(\mathcal{B}_n)''}}, \widehat{\psi_2 \upharpoonright_{\pi_0(\mathcal{B}_n)''}}) \\ &= \psi_1(e_n) \log \frac{\psi_1(e_n)}{\psi_2(e_n)} + \psi_1(1 - e_n) \log \frac{\psi_1(1 - e_n)}{\psi_2(1 - e_n)}. \end{aligned}$$

The above estimate contradict to the assumption. ■

Finally, we recall the continuity of the relative entropy.

Lemma 4.10 ([22, Corollary 5.12 (i)]) *Let ψ_i, ψ, ϕ_i and ϕ be normal states on a von Neumann algebra \mathfrak{M} . If ψ_i and ϕ_i converge to ψ and ϕ in $\sigma(\mathfrak{M}_*, \mathfrak{M})$ topology, respectively, then*

$$S(\psi, \phi) \leq \liminf_i S(\psi_i, \phi_i). \quad (4.10)$$

5 KMS states on the resolvent CCR algebra

In this section, we consider KMS states on the resolvent CCR algebra.

In our model, the time evolution $\alpha_t(Q)$ is not be norm continuous as a function of t for certain Q . However, the set of elements Q for which $\alpha_t(Q)$ have analytic extension as functions of t is weakly dense in regular representations. We introduce the notion of KMS states in the following manner.

Definition 5.1 Let $\alpha_t, t \in \mathbf{R}$, be a (not necessarily continuous) one-parameter group of $*$ -automorphism on a unital C^* -algebra \mathcal{A} . The state ψ is an (α, β) -KMS state, if ψ is α invariant state, i.e. $\psi(\alpha_t(Q)) = \psi(Q)$, $Q \in \mathcal{A}$, and $\psi(Q\alpha_t(R))$ is a continuous function in $t \in \mathbf{R}$ for any $Q, R \in \mathcal{A}$ satisfying the KMS boundary condition, namely, there exists a function $F_{Q,R}(t)$ holomorphic in I_β , bounded continuous on the closure of I_β such that

$$F_{Q,R}(t) = \psi(Q\alpha_t(R)), \quad F_{Q,R}(t + i\beta) = \psi(\alpha_t(R)Q) \quad (5.1)$$

for any $Q, R \in \mathcal{A}$.

5.1 KMS state associated with the weakly coupled anharmonic oscillators

In this subsection, we consider KMS states on the resolvent CCR algebra associated with the anharmonic dynamics defined in (3.2).

To begin with, we recall for our resolvent CCR algebra there exists the trivial (not interesting) state $\psi_{trivial}$ defined by

$$\psi_{trivial}(R(\lambda_1, f_1)R(\lambda_2, f_2) \cdots R(\lambda_k, f_k)) = 0 \quad (5.2)$$

for any $\lambda_i \in \mathbf{C}$ and f_j .

For any finite system, we have decomposition of the KMS state into the regular part and the singular part.

Lemma 5.2 We identify \mathfrak{R}_L with operators in the Schrödinger representation on $\mathfrak{H}_{\Lambda_L} = L^2(\mathbf{R}^{2L})$ and Let \mathcal{K}_L be the algebra of compact operators on $L^2(\mathbf{R}^{2L})$ which we regard as a sub-algebra of \mathfrak{R}_L . Let H be a positive self-adjoint operator on \mathfrak{H}_{Λ_L} satisfying the following conditions. :

(a) $e^{itH}\pi_{\Lambda_L}(Q)e^{-itH}$ ($Q \in \mathfrak{R}_L$) gives rise to a one-parameter group of automorphisms of \mathfrak{R}_L denoted by $\alpha_t(Q)$. $\pi_{\Lambda_L}(\alpha_t(Q)) = e^{itH}\pi_{\Lambda_L}(Q)e^{-itH}$

(b) $e^{-\beta H}$ is a trace class operator on \mathfrak{H}_{Λ_L} .

Let ψ_β be a β -KMS state for α_t . There exist β -KMS states ψ_s and ψ_r satisfying the following properties.

(i) The kernel of the GNS representation for ψ_s contains the compact operator algebra \mathcal{K}_L on \mathfrak{H}_{Λ_L} .

(ii) ψ_r is the regular KMS state defined by

$$\psi_r(Q) = \frac{\text{tr}_{\mathfrak{H}_{\Lambda_L}}(e^{-\beta H}Q)}{\text{tr}_{\mathfrak{H}_{\Lambda_L}}(e^{-\beta H})}, \quad Q \in \mathfrak{R}_L. \quad (5.3)$$

(iii) ψ_β is a convex combination of ψ_s and ψ_r ,

$$\psi_\beta = \lambda\psi_r + (1 - \lambda)\psi_s$$

for some a positive real number λ $0 \leq \lambda \leq 1$

Proof. Let p_j ($j = 0, 1, 2, \dots$) be the mutually orthogonal projections in \mathfrak{R}_L such that $\pi_{\Lambda_L}(p_j)$ is the rank one projection associated with an eigenvector for an eigenvalue ϵ_j of H and

$$H = \sum_j \epsilon_j \pi_{\Lambda_L}(p_j).$$

Set $P_m = \sum_{j=1}^n p_j$ and $P = w - \lim_{m \rightarrow \infty} \pi_{\psi_\beta}(P_m)$ on the GNS representation associated with ψ_β . We claim that the projection P is in the centre of the von Neumann algebra $\pi_{\psi_\beta}(\mathfrak{R}_L)''$. In fact, by definition P commutes with any elements in $\pi_{\psi_\beta}(\mathcal{K}_L)$ and for any $Q \in \mathfrak{R}_L$ QP_m is of finite rank in the Schrödinger representation, $\pi_{\psi_\beta}(QP_m)$ and its weak limit commutes with $\pi_{\psi_\beta}(\mathfrak{R}_L)$.

Set $\lambda = \lim_{m \rightarrow \infty} \psi(P_m)$ and

$$\psi_r(Q) = \lim_{m \rightarrow \infty} \psi_\beta(QP_m), \quad \psi_s(Q) = \lim_{m \rightarrow \infty} \psi_\beta(Q(1 - P_m))$$

for any $Q \in \mathfrak{R}_L$. As P is in the centre of $\pi_{\psi_\beta}(\mathfrak{R}_L)''$, ψ_r and ψ_s are β -KMS states.

For any compact $Q \in \mathcal{K}_L$, $\psi_s(Q^*Q) = \lim_{m \rightarrow \infty} \psi_\beta(Q^*Q(1 - P_m)) = 0$ as $\{P_m\}$ is an approximate unit for $Q \in \mathcal{K}_L$. As the GNS vector of the KMS state ψ_s is separating for $\pi_{\psi_s}(\mathfrak{R}_L)$ the kernel of π_{ψ_s} contains \mathcal{K}_L , $\pi_{\psi_s}(\mathcal{K}_L) = 0$.

Now suppose that $\lambda = \lim_{m \rightarrow \infty} \psi(P_m) \neq 0$. Then, $\psi(p_j) = \psi_r(p_j) \neq 0$ for any j . As λ does not vanish, there is at least on j satisfying $\psi_r(p_j) \neq 0$. On the other hand, for a matrix unit system p_{ij} of \mathcal{K}_L satisfying $p_{ij}p_{ji} = p_i$, the KMS condition implies

$$\psi_r(p_i) = \psi_r(p_{ij}p_{ji}) = e^{\epsilon_j - \epsilon_i} \psi_r(p_j), \quad \psi_r(p_{kl}) = 0 (k \neq l)$$

which shows that $\psi_r(p_j) \neq 0$ for any j .

These equation tells us that (5.3) for $Q = AP_m$ ($A \in \mathfrak{R}_L$). By taking the limit $m \rightarrow \infty$, we obtain (5.3) holds for any $Q \in \mathfrak{R}_L$. ■

Lemma 5.3 *For any $\beta > 0$, if ψ is a $(\alpha_t^{free,1}, \beta)$ -KMS state of a single harmonic oscillator, $\psi_s = \psi_{trivial}$ where $\psi_{trivial}$ is defined in (5.2).*

Proof. Let φ be a KMS state for $\alpha_t^{free,1}$ such that the kernel of the GNS representation for φ contains the compact operator algebra. Let $\{\pi_\varphi(\cdot), \Omega, \mathfrak{H}\}$ be the GNS triple associated with φ .

Note that the quotient $\tilde{\mathfrak{R}}_1 = \mathfrak{R}_1/\mathcal{K}_1$ is .

Assuming $\pi_\varphi(Q) = 0, Q \in \mathcal{K}_L$, we show $\pi_{varphi}(R(\lambda, f)) = 0$. If $\pi_\varphi(Q) = 0, Q \in \mathcal{K}_L$, φ gives rise to the KMS state $\tilde{\varphi}$ of the quotient algebra $\tilde{\mathfrak{R}} = \mathfrak{R}_1/\mathcal{K}_1$ for the time evolution $\tilde{\alpha}_t^{free,1}$ induced by $\alpha_t^{free,1}$.

Let Q and R be entire analytic elements in \tilde{R} . Due to the KMS boundary condition and commutativity of \tilde{R} , $\tilde{\varphi}(Q\sigma_t(R))$ is bounded on the whole complex plane and is entire, so $\tilde{\varphi}(Q\sigma_t(R))$ is a constant ,

$$\tilde{\varphi}(Q\sigma_t(R)) = \tilde{\varphi}(QR).$$

We set $Q = R = \pi_\varphi(f(x))$ where f is a real continuous function with one variable vanishing at infinity . (x is the position operator.) As $\alpha_t^{free,L}(f(x)) = f(\cos \omega t \hat{x} + \sin \omega t \hat{p})$, for $t = \pi/(2\omega)$ $f(x)\alpha_t^{free,L}(f(x))$ is a compact operator. Thus,

$$\varphi(f(x)^2) = 0.$$

As Ω is separating for $\pi_\varphi(\tilde{R})'$,

$$\pi_\varphi(f(x)) = 0.$$

It turns out

$$\pi_\varphi(\alpha_t^{free,L}(f(x))) = \pi_\varphi(f(\cos \omega t \hat{x} + \sin \omega t \hat{p}))$$

which shows that $\varphi = \psi_{trivial}$. ■

The above lemma shows any KMS state for an inner perturbation of a single harmonic oscillator .

Lemma 5.4 *We consider the quantum mechanical system with one degree of freedom \mathfrak{R}_1 , and suppose $H = p^2 + x^2 + V$ ($\in \mathfrak{R}_1$ gives rise to the generator of the time evolution α_t of \mathfrak{R}_1 . If $\beta > 0$ and if ψ is a (α_t, β) -KMS state*

$$\psi = \lambda \psi_r + (1 - \lambda) \psi_{trivial}$$

for some λ with $0 \leq \lambda \leq 1$

Proof. The perturbed ψ^{-V} is quasi-equivalent to a KMS state φ of the free time evolution $\alpha_t^{free,L}$ for which the claim of the lemma is valid. As $\psi_{trivial}^V = \psi_{trivial}$ we obtain our claim ■

We are not certain that $\psi_s = \psi_{trivial}$ holds for more general finite quantum systems, though, the physical meaning of singular KMS states is clear and in what follows, we shall consider regular KMS states.

Lemma 5.5 *For any positive integers $L < L'$ and any positive function $F \in \otimes_{k \in \Lambda_L} L^\infty(\mathbf{R}, dx_k)$ the following estimates are valid:*

$$e^{-2\beta\|\varphi\|_\infty} \text{Tr}_{L' \setminus L}(e^{-\beta H_{L' \setminus L}}) \text{Tr}_L(e^{-\beta H_L} M_F) \leq \text{Tr}_{L'}(e^{-\beta H_{L'}} M_F) \quad (5.4)$$

$$\text{Tr}_{L'}(e^{-\beta H_{L'}} M_F) \leq e^{2\beta\|\varphi\|_\infty} \text{Tr}_{L' \setminus L}(e^{-\beta H_{L' \setminus L}}) \text{Tr}_L(e^{-\beta H_L} M_F) \quad (5.5)$$

where $H_{L' \setminus L} = H(\Lambda_{L'} \setminus \Lambda_L)$, M_F is the multiplication operator of F on \mathfrak{H}_{Λ_L} and $\|\varphi\|_\infty$ is the supremum norm of φ .

Proof. Note that for $\beta > 0$, $e^{-\beta H_L}$ is a trace class operator on \mathfrak{H}_{Λ_L} and also a Hilbert-Schmidt class operator. Thus, $e^{-\beta H_L}$ has the integral kernel $e^{-\beta H_L}(x, y)$ satisfying $\int_{\mathbf{R}^{|\Lambda_L|}} \int_{\mathbf{R}^{|\Lambda_L|}} |e^{-\beta H_L}(x, y)|^2 dx dy < \infty$.

For $L < L'$, we have

$$\begin{aligned}
\Upsilon_{L'} &= \sum_{\Lambda \subset \Lambda_L} \Phi(\Lambda) = \sum_{k \in \Lambda_{L'}} V(x_k) + \sum_{k, k+1 \in \Lambda_{L'}} \varphi(x_k - x_{k+1}) \\
&\geq \sum_{k \in \Lambda_{L'}} V(x_k) + \sum_{k, k+1 \in \Lambda_L} \varphi(x_k - x_{k+1}) + \sum_{k, k+1 \in \Lambda_{L'} \setminus \Lambda_L} \varphi(x_k - x_{k+1}) - 2 \|\varphi\|_\infty \\
&= \sum_{\Lambda \subset \Lambda_{L'}} \Phi(\Lambda) + \sum_{\Lambda \subset \Lambda_{L'} \setminus \Lambda_L} \Phi(\Lambda) - 2 \|\varphi\|_\infty \\
&= \Upsilon_L + \Upsilon_{L' \setminus L} - 2 \|\varphi\|_\infty,
\end{aligned}$$

and

$$\Upsilon_{L'} \leq \Upsilon_L + \Upsilon_{L' \setminus L} + 2 \|\varphi\|_\infty.$$

Note that $e^{-\beta H_{L'}^h}$ has the Mehler kernel $k_\beta^h(x, y) \in \mathcal{S}(\mathbf{R}^{2L'})$,

$$k_\beta^h(x, y) = \left(\frac{\omega}{2\pi \sinh(2\omega\beta)} \right)^{\frac{n}{2}} \prod_{k \in \Lambda_{L'}} \exp \left(- \frac{\omega(x_k^2 + y_k^2) \coth(2\omega\beta) - 2 \operatorname{cosech}(2\omega\beta) x_k y_k}{2} \right)$$

for $x = (x_{-L'+1}, \dots, x_{-1}, x_0, x_1, \dots, x_{L'})$, $y = (y_{-L'+1}, \dots, y_{L'}) \in \mathbf{R}^{2L'}$. For any positive functions $f, g \in \mathcal{S}(\mathbf{R}^{2L'})$, we have

$$\begin{aligned}
&\left\langle f, (e^{-\frac{\beta H_{L'}^h}{n}} e^{-\frac{\beta \Upsilon_{L'}}{n}})^n g \right\rangle_{L^2} \\
&= \int_{\mathbf{R}^{2L'}} f(w) \int_{\mathbf{R}^{2L'}} k_{\beta/n}^h(w, z_1) e^{-\frac{\beta \Upsilon_{L'}}{n}}(z_1) \int_{\mathbf{R}^{2L'}} k_{\beta/n}^h(z_1, z_2) e^{-\frac{\beta \Upsilon_{L'}}{n}}(z_2) \\
&\quad \times \dots \times \int_{\mathbf{R}^{2L'}} k_{\beta/n}^h(z_{n-1}, z_n) e^{-\frac{\beta \Upsilon_{L'}}{n}}(z_n) g(z_n) dz_n \dots dz_2 dz_1 dw \\
&\geq e^{-2\beta \|\varphi\|_\infty} \int_{\mathbf{R}^{2L'}} f(w) \int_{\mathbf{R}^{2L'}} k_{\beta/n}^h(w, z_1) e^{-\beta \frac{\Upsilon_L + \Upsilon_{L' \setminus L}}{n}}(z_1) \\
&\quad \times \dots \times \int_{\mathbf{R}^{2L'}} k_{\beta/n}^h(z_{n-1}, z_n) e^{-\beta \frac{\Upsilon_L + \Upsilon_{L' \setminus L}}{n}}(z_n) g(z_n) dz_n \dots dz_2 dz_1 dw \\
&= e^{-2\beta \|\varphi\|_\infty} \left\langle f, (e^{-\frac{\beta H_{L'}^h}{n}} e^{-\beta \frac{\Upsilon_L + \Upsilon_{L' \setminus L}}{n}})^n g \right\rangle_{L^2}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\langle f, e^{-\beta H_{L'}} g \rangle_{L^2} &= \lim_{n \rightarrow \infty} \left\langle f, (e^{-\frac{\beta H_{L'}^h}{n}} e^{-\frac{\beta \Upsilon_{L'}}{n}})^n g \right\rangle_{L^2} \\
&\geq \lim_{n \rightarrow \infty} e^{-2\beta \|\varphi\|_\infty} \left\langle f, (e^{-\frac{\beta H_{L'}^h}{n}} e^{-\frac{\beta \Upsilon_L + \beta \Upsilon_{L' \setminus L}}{n}})^n g \right\rangle_{L^2} \\
&= e^{-2\beta \|\varphi\|_\infty} \langle f, e^{-\beta H_L - \beta H_{L' \setminus L}} g \rangle_{L^2}.
\end{aligned}$$

This means that

$$e^{-\beta H_{L'}}(x, y) \geq e^{-2\beta \|\varphi\|_\infty} e^{-\beta H_L - \beta H_{L' \setminus L}}(x, y), \quad x, y \in \mathbf{R}^{2L'}.$$

Since $e^{-\beta H_{L'}}$ is a trace class operator, the integral kernel of $e^{-\beta H_{L'}}$ satisfies $\int_{\mathbf{R}^{2L'}} e^{-\beta H_{L'}}(x, x) dx < \infty$. Thus, we obtain the following estimates for any positive function $F \in \bigotimes_{k \in \Lambda_L} L^\infty(\mathbf{R})$:

$$\begin{aligned} \text{Tr}_{L'}(e^{-\beta H_L} M_F) &= \int_{\mathbf{R}^{2L}} e^{-\beta H_L}(x, x) F(x) dx \\ &\geq e^{-2\beta \|\varphi\|_\infty} \int_{\mathbf{R}^{2L'}} e^{-\beta(H_{L' \setminus L} + H_L)}(x, x) F(x) dx \\ &= e^{-2\beta \|\varphi\|_\infty} \text{Tr}_{L'}(e^{-\beta H_{L' \setminus L}} e^{-\beta H_L} M_F) \\ &= e^{-2\beta \|\varphi\|_\infty} \text{Tr}_{L' \setminus L}(e^{-\beta H_{L' \setminus L}}) \text{Tr}_L(e^{-\beta H_L} M_F) \end{aligned}$$

and

$$\begin{aligned} \text{Tr}_{L'}(e^{-\beta H_{L'}} M_F) &\leq e^{2\beta \|\varphi\|_\infty} \text{Tr}_{L'}(e^{-\beta H_{L' \setminus L}} e^{-\beta H_L} M_F) \\ &= e^{2\beta \|\varphi\|_\infty} \text{Tr}_{L' \setminus L}(e^{-\beta H_{L' \setminus L}}) \text{Tr}_L(e^{-\beta H_L} M_F). \end{aligned}$$

Thus, we obtain (5.4) and (5.5). ■

Proposition 5.6 *For any positive integers $L \leq L'$ and any $F \in \mathfrak{R}_L$ such that $\pi_0(F)$ is a positive multiplication operator on \mathfrak{H}_{Λ_L} , the following estimate hold:*

$$e^{-4\beta \|\varphi\|_\infty} \psi_L(F) \leq \psi_{L'}(F) \leq e^{4\beta \|\varphi\|_\infty} \psi_L(F), \quad (5.6)$$

where ψ_L and $\psi_{L'}$ are states on \mathfrak{R}_L and $\mathfrak{R}_{L'}$ defined in (??), respectively.

Proof. By (5.4) and (5.5), we obtain the following inequalities:

$$\begin{aligned} e^{-2\beta \|\varphi\|_\infty} \text{Tr}_{L \setminus L'}(e^{-\beta H_{L \setminus L'}}) \text{Tr}_{L'}(e^{-\beta H_{L'}} M_F) &\leq \text{Tr}_L(e^{-\beta H_L} M_F), \\ \text{Tr}_L(e^{-\beta H_L} M_F) &\leq e^{2\beta \|\varphi\|_\infty} \text{Tr}_{L \setminus L'}(e^{-\beta H_{L \setminus L'}}) \text{Tr}_{L'}(e^{-\beta H_{L'}} M_F), \\ e^{-2\beta \|\varphi\|_\infty} \text{Tr}_{L \setminus L'}(e^{-\beta H_{L \setminus L'}}) \text{Tr}_{L'}(e^{-\beta H_{L'}}) &\leq \text{Tr}_L(e^{-\beta H_L}), \\ \text{Tr}_L(e^{-\beta H_L}) &\leq e^{2\beta \|\varphi\|_\infty} \text{Tr}_{L \setminus L'}(e^{-\beta H_{L \setminus L'}}) \text{Tr}_{L'}(e^{-\beta H_{L'}}). \end{aligned}$$

Thus, we obtain (5.6). ■

Note that ψ_L is also a state on \mathcal{W}_L , i.e.

$$\psi_L(W) = \frac{\text{Tr}_L(e^{-\beta H_L} \pi_0(W))}{\text{Tr}_L(e^{-\beta H_L})}, \quad W \in \mathcal{W}_L. \quad (5.7)$$

Also, for the Weyl CCR algebra the following statement follows.

Proposition 5.7 *For any positive integers $L \leq L'$ and any $F \in \mathcal{W}_L$ such that $\pi_0(F)$ is a positive multiplication operator on \mathfrak{H}_{Λ_L} , the following estimate hold:*

$$e^{-4\beta \|\varphi\|_\infty} \psi_L(F) \leq \psi_{L'}(F) \leq e^{4\beta \|\varphi\|_\infty} \psi_L(F) \quad (5.8)$$

where ψ_L and $\psi_{L'}$ are states on \mathcal{W}_L and $\mathcal{W}_{L'}$ defined in (5.7), respectively.

Since $e^{-\beta(p^2+\omega^2x^2)}$ is a trace class operator on $L^2(\mathbf{R}, dx)$ and by [6, Proposition 5.2.27], $e^{-\beta d\Gamma(p^2+\omega^2x^2)}$ is a trace class on $\mathcal{F}_+(L^2(\mathbf{R}, dx))$, where $d\Gamma(p^2 + \omega^2x^2)$ is the second quantization of $p^2 + \omega^2x^2$ and $\mathcal{F}_+(L^2(\mathbf{R}, dx))$ is the Bose-Fock space of $L^2(\mathbf{R}, dx)$ (See also [6, Section 5.2.1]). Put

$$\tilde{\psi}_L := \psi_L \otimes \frac{\text{Tr}_{\mathcal{F}_+(L^2(\mathbf{R}, dx))}(e^{-\beta H^h(\mathbf{Z} \setminus \Lambda_L)} \pi_0(\cdot))}{\text{Tr}_{\mathcal{F}_+(L^2(\mathbf{R}, dx))}(e^{-\beta H^h(\mathbf{Z} \setminus \Lambda_L)})},$$

then $\tilde{\psi}_L$ is a regular state on \mathcal{W} and $\tilde{\psi}_L \upharpoonright_{\mathcal{W}_L} = \psi_L$. Thus, the regular state ψ_L on \mathcal{W}_L can extend to a regular state on \mathcal{W} . Since \mathcal{W} is a unital C^* -algebra, the family of states $\{\tilde{\psi}_L\}_{L \in \mathbf{N}}$ has at least one cluster point ψ . Next, we show that ψ is a regular state.

Theorem 5.8 *The state ψ defined in the above is a regular state on \mathcal{W} .*

Proof. To show the regularity of ψ , we show that for t in $|t| \leq \delta$

$$\tilde{\psi}_L(QW_0(t))$$

is equicontinuous with respect to L where $Q = Q(x)$ is an arbitrary essentially bounded function on \mathbf{R}^{2L} and $W_0(t) = e^{itp_0}$. (We can show the continuity of $\lim_L \tilde{\psi}_L(QW_0(t))$ for the general $W = e^{i \sum_{k=-L+1}^L t_k p_k}$ in the same way.)

For simplicity of presentation, we consider the case $\omega = 1$ here. Note that

$$\begin{aligned} \tilde{\psi}_L(QW_0(t)) &= \\ \frac{1}{Z_{\beta L}^2} \int \int e^{-\beta/2 H_L(x, y)} Q(x) \frac{e^{-\beta/2 H_L(x + t^{(0)}, y)}}{e^{-\beta/2 H_L(x, y)}} e^{-\beta/2 H_L(x, y)} dx dy \end{aligned}$$

where $Z_{\beta L}^2$ is the normalization constant

$$Z_{\beta L}^2 = \int \int (e^{-\beta/2 H_L(x, y)})^2 dx dy = \int e^{-\beta H_L(x, x)} dx.$$

and $x + t^{(0)}$ is the addition of t to x at the component corresponding to the origin of the integer lattice \mathbf{Z} .

For $x = (\cdots, x_{-1}, x_0, x_1, \cdots)$ and $y = (\cdots, y_{-1}, y_0, y_1, \cdots)$ we claim that

$$e^{-c(t)} \frac{k_{\beta}^h(x_0 + t, y_0)}{k_{\beta}^h(x_0, y_0)} \leq \frac{e^{-\beta/2 H_L(x + t^{(0)}, y)}}{e^{-\beta/2 H_L(x, y)}} \leq e^{c(t)} \frac{k_{\beta}^h(x_0 + t, y_0)}{k_{\beta}^h(x_0, y_0)} \quad (5.9)$$

where

$$\begin{aligned} c(t) &= \sup_{x_0} |V(x_0 + t) - V(x_0)| + \sup_{x_0, x_1} |\varphi(x_0 - x_1 + t) - \varphi(x_0 - x_1)| \\ &\quad + \sup_{x_0, x_{-1}} |\varphi(x_{-1} - x_0 + t) - \varphi(x_{-1} - x_0)|. \end{aligned}$$

$\lim_{t \rightarrow 0} c(t) = 0$ due to uniform continuity of V and φ and this bound implies regularity.

We now show (5.9). Note the following tautological equalities holds. For any $n \in \mathbb{N}$,

$$\begin{aligned} k_\beta^h(x_0 + t, y_0) &= \int \cdots \int k_{\beta/n}^h(x_0 + t, z_1) k_{\beta/n}^h(z_1, z_2) \cdots k_{\beta/n}^h(z_n, y_0) dz_1 \cdots dz_n \\ &= \int \cdots \int k_{\beta/n}^h(x_0 + t, z_1 + s_1) k_{\beta/n}^h(z_1 + s_1, z_2 + s_2) \cdots k_{\beta/n}^h(z_n + s_n, y_0) dz_1 \cdots dz_n \end{aligned} \quad (5.10)$$

for any constants s_k . Then, up to a multiplicative factor, \tilde{C}_{nt} ,

$$\begin{aligned} &k_{\beta/n}^h(x_0 + t, z_1 + s_1) k_{\beta/n}^h(z_1 + s_1, z_2 + s_2) \cdots k_{\beta/n}^h(z_n + s_n, y_0) = \\ &\tilde{C}_{nt} \times \exp\left[-\frac{1}{2 \sinh(2\beta/n)} \sum_{k=0}^n \{\cosh(2\beta/n)((z_k + s_k)^2 + (z_{k+1} + s_{k+1})^2) \right. \\ &\quad \left. - 2(z_k + s_k)(z_{k+1} + s_{k+1})\}\right] \end{aligned} \quad (5.11)$$

where we set $z_0 = x, z_{n+1} = y$. In the exponent, we can write

$$\begin{aligned} &\sum_{k=0}^n \{\cosh(2\beta/n)((z_k + s_k)^2 + (z_{k+1} + s_{k+1})^2) - 2(z_k + s_k)(z_{k+1} + s_{k+1})\} \\ &= \sum_{k=0}^n \{\cosh(2\beta/n)(z_k^2 + z_{k+1}^2) - 2z_k z_{k+1}\} \\ &+ \sum_{k=0}^{n+1} A_{n,k}(s) z_k + \Sigma_n(t, x_0, y_0) \end{aligned} \quad (5.12)$$

where $A_{n,k}(s)$ (linear in s_k) and $\Sigma_n(t, x_0, y_0)$ (quadratic in s_k) are terms independent on z_k . Now we choose the constants s_k satisfying the condition $A_{n,k}(s) = 0$ $s_0 = t$ $s_{n+1} = 0$. We do not need the exact form of $A_{n,k}(s)$ and $\Sigma_n(s, t)$ here but what we need are bounds $|s_k| \leq \tilde{A}|t|$ independent of n .

Thus, we obtain

$$\begin{aligned} k_\beta^h(x_0 + t, y_0) &= \exp\left[-\frac{\Sigma_n(t, x_0, y_0)}{2 \sinh(2\beta/n)}\right] \times \\ &\int \cdots \int k_{\beta/n}^h(x_0, z_1) k_{\beta/n}^h(z_1, z_2) \cdots k_{\beta/n}^h(z_n, y_0) dz_1 \cdots dz_n \end{aligned} \quad (5.13)$$

and

$$\lim_{n \rightarrow \infty} \exp\left[-\frac{\Sigma_n(t, x_0, y_0)}{2 \sinh(2\beta/n)}\right] = \frac{k_\beta^h(x_0 + t, y_0)}{k_\beta^h(x_0, y_0)} \quad (5.14)$$

To show (5.9) we apply the Trotter-Kato formula again to

$$e^{-\beta H_L^h}(x + t^{(0)}, y) = \lim_{n \rightarrow \infty} (e^{-\frac{\beta}{n} H_L^h} e^{-\frac{\beta \tau_L}{n}})^n(x + t^{(0)}, y) \quad (5.15)$$

We consider now $(e^{-\frac{\beta H_L^h}{n}} e^{-\frac{\beta \Upsilon_L}{n}})^n (x + t^{(0)}, y)$ at each n in (5.15). The integral kernel of $(e^{-\frac{\beta H_L^h}{n}} e^{-\frac{\beta \Upsilon_L}{n}})^n$ is an iteration of integral in which the shift $x + t^{(0)}$ of the variable affect only to the integral associated to the particle at the origin and its nearest neighbor. In that integral, we denote the variable at the site -1 at the lattice by $z_k^{(-1)}$ and that at the site 1 at the lattice by $z_k^{(1)}$. Then, the contribution to the iterated integral from the origin in $(e^{-\frac{\beta H_L^h}{n}} e^{-\frac{\beta \Upsilon_L}{n}})^n (x, y)$ is

$$\begin{aligned} \int & \cdots \int k_{\beta/n}^h(x_0, z_1) \exp \left[-\frac{\beta}{n} (V(z_1) + \varphi(z_1 - z_1^{(-1)}) + \varphi(z_1^{(1)} - z_1)) \right] \\ & \times k_{\beta/n}^h(z_1, z_2) \exp \left[-\frac{\beta}{n} (V(z_2) + \varphi(z_2 - z_2^{(-1)}) + \varphi(z_2^{(1)} - z_2)) \right] \\ & \cdots k_{\beta/n}^h(z_n, y_0) \exp \left[-\frac{\beta}{n} (V(y_0) + \varphi(y_0 - y_{-1}) + \varphi(y_1 - y_0)) \right] dz_1 \cdots dz_n \end{aligned} \quad (5.16)$$

After the shift of variable $z_k \rightarrow z_k + s_k$ as in (5.10) the corresponding integral for $(e^{-\frac{\beta H_L^h}{n}} e^{-\frac{\beta \Upsilon_L}{n}})^n (x + t^{(0)}, y)$ is

$$\begin{aligned} \int & \cdots \int k_{\beta/n}^h(x_0, z_1) \exp \left[-\frac{\beta}{n} (V(z_1) + \varphi(z_1 - s_1 - z_1^{(-1)}) + \varphi(z_1^{(1)} - z_1 + s_1)) \right] \\ & \times k_{\beta/n}^h(z_1, z_2) \exp \left[-\frac{\beta}{n} (V(z_2 - s_2) + \varphi(z_2 - s_2 - z_2^{(-1)}) + \varphi(z_2^{(1)} - z_2 + s_2)) \right] \\ & \cdots dz_1 \cdots dz_n \times \exp \left[-\frac{\Sigma_n(t, x_0, y_0)}{2 \sinh(2\beta/n)} \right] \end{aligned} \quad (5.17)$$

Then,

$$\begin{aligned} (5.16) & \times e^{-c(\tilde{A}t)} \exp \left[-\frac{\Sigma_n(t, x_0, y_0)}{2 \sinh(2\beta/n)} \right] \\ & \leq (5.17) \leq (5.16) \times e^{c(\tilde{A}t)} \exp \left[-\frac{\Sigma_n(t, x_0, y_0)}{2 \sinh(2\beta/n)} \right] \end{aligned} \quad (5.18)$$

By taking the limit $n \rightarrow \infty$ we obtain the bound (5.9).

Finally we can show the regularity of ψ by using (5.9), Proposition 5.6 and the Lebesgue dominated convergence theorem. ■

The following proposition corresponds to the Gibbs condition. We consider the perturbation of a regular state and the automorphism α defined in (3.2). The perturbation of an automorphism and a state on a C*-algebra or a von Neumann algebra is defined in [6, Proposition 5.4.1] and [6, Theorem 5.4.4].

Proposition 5.9 *Let ϕ be a regular (α, β) -KMS state on \mathfrak{A} , where α is an automorphism defined in (3.2) and $\beta > 0$. Put $W(L) := \pi_0^{-1}(\Phi(\{L, L+1\})) +$*

$\pi_0^{-1}(\Phi(\{-L, -L+1\}))$, $L \in \mathbf{N}$. Then ϕ satisfies the following condition:

$$\phi^{\beta W(L)} = \psi_L \otimes \tilde{\phi} \quad (5.19)$$

for all $L \in \mathbf{N}$, where $\tilde{\phi}$ is a state over \mathfrak{R}_{L^c} , $\phi^{\beta W(L)}$ is a perturbed state of ϕ by $\beta W(L)$.

Proof. For positive integers $L < L'$, let $\gamma_t^{L',L}$ be the perturbed automorphism of $\alpha_t^{L'}$ by $\beta W(L)$. Since $H_{L'} - \pi_0(W(L)) = H_{L' \setminus L} + H_L$ and H_L and $H_{L' \setminus L}$ are commute, $\gamma_t^{L',L} = \alpha_t^L \otimes \alpha_t^{L' \setminus L}$. The automorphism $\alpha_t^{L' \setminus L}$ converges strongly to an automorphism $\alpha_t^{L^c}$ on \mathfrak{R}_{L^c} when $L' \rightarrow \infty$ by Theorem 3.3. Note that the perturbed state $\phi^{\beta W(L)}$ is a (γ, β) -KMS state by construction, where $\gamma_t = \alpha_t^L \otimes \alpha_t^{L^c}$.

For $0 < R \in \mathfrak{R}_{L^c}$, we define the state $\phi_R^{\beta W(L)}$ on \mathfrak{R}_L by

$$\phi_R^{\beta W(L)}(Q) = \frac{\phi^{\beta W(L)}(QR)}{\phi^{\beta W(L)}(R)}, \quad Q \in \mathfrak{R}_L.$$

Note that $\phi_R^{\beta W(L)}$ is a regular state by construction and by $\gamma_t = \alpha_t^L \otimes \alpha_t^{L^c}$, $\phi_R^{\beta W(L)}$ is an (α^L, β) -KMS state. By Lemma 5.4, $\phi_R^{\beta W(L)} = \psi_L$. Thus, for all $Q \in \mathfrak{R}_L$ and $0 < R \in \mathfrak{R}_{L^c}$

$$\phi^{\beta W(L)}(QR) = \psi_L(Q) \phi^{\beta W(L)}(R). \quad (5.20)$$

For any self-adjoint element $R \in \mathfrak{R}_{L^c}$ and any $\varepsilon > 0$, $R + (\|R\| + \varepsilon)\mathbb{1}$ is a strictly positive operator. Then we obtain

$$\phi^{\beta W(L)}(Q(R + (\|R\| + \varepsilon)\mathbb{1})) = \psi_L(Q) \phi^{\beta W(L)}(R + (\|R\| + \varepsilon)\mathbb{1}).$$

Since for any element $R \in \mathfrak{R}_{L^c}$ can decompose two self-adjoint elements R_1 and R_2 such that $R = R_1 + iR_2$. By the linearity of $\phi^{\beta W(L)}$, the equation (5.20) holds for any elements $Q \in \mathfrak{R}_L$ and $R \in \mathfrak{R}_{L^c}$. Thus,

$$\phi^{\beta W(L)} = \psi_L \otimes \phi^{\beta W(L)} \upharpoonright_{\mathfrak{R}_{L^c}}.$$

Put $\tilde{\phi} = \phi^{\beta W(L)} \upharpoonright_{\mathfrak{R}_{L^c}}$, then we get the claim. ■

Remark 5.10 For $Q \in \mathfrak{R}$, it may not be a linear combination of the form of $A \otimes B$ for $A \in \mathfrak{R}_L$ and $B \in \mathfrak{R}_{L^c}$. However, by Lemma 4.3 and [8, Theorem 4.2. (v)], for a regular state ϕ on \mathfrak{R} and any positive integer L , $\pi_\phi(\mathfrak{R}_L) \otimes \pi_\phi(\mathfrak{R}_{L^c})$ is a weakly dense sub-algebra in $\pi_\phi(\mathfrak{R})$. For $Q \in \mathfrak{R}$, there exists a positive integers $\{L(n)\}_{n \in \mathbf{N}}$ such that $L \leq L(n)$ for any $n \in \mathbf{N}$ and a sequence $\sum_i a_i^{(n)} R_i^{(n)} \otimes K_i^{(n)}$ such that $a_i^{(n)} \in \mathbf{C}$, $R_i^{(n)} \in \mathfrak{R}_L$, $K_i^{(n)} \in \mathfrak{K}(\Lambda_{L(n)} \setminus \Lambda_L)$ and

$$\pi_\phi(Q) = \text{w-lim}_n \sum_i a_i^{(n)} \pi_\phi(A_i^{(n)}) \otimes \pi_\phi(K_i^{(n)}).$$

and we can defined the product state $\psi_L \otimes \phi^{\beta W(L)} \upharpoonright_{\mathfrak{R}_{L^c}}$ for any $Q \in \mathfrak{R}$.

Finally, we show uniqueness of (α, β) -KMS state for $\beta > 0$ in Theorem 1.3, . Due to Theorem 5.8, ψ gives rise to a regular state on \mathcal{W} and hence a regular state of \mathfrak{R} .

5.2 Proof of Theorem 1.3.

First, we show $\psi(Q\alpha_t(R))$ is continuous in $t \in \mathbf{R}$ for any $Q, R \in \mathfrak{R}$. Since $\bigcup_{L \in \mathbf{N}} \mathfrak{R}_L$ is norm dense in \mathfrak{R} and $\mathfrak{R}_L \subset \mathfrak{R}_{L'}$ whenever $L \leq L'$, we show $\psi(Q\alpha_t(R))$ is continuous in $t \in \mathbf{R}$ for any $Q, R \in \mathfrak{R}_L$. By Theorem 3.3, Proposition 4.4 and Theorem 5.8, for any positive integer L and any $Q, R \in \mathfrak{R}_L$, $\psi(Q\alpha_t(R))$ is continuous in $t \in \mathbf{R}$. In fact, for any $\varepsilon > 0$, there exists a positive integer L such that $\|\alpha_t(R) - \alpha_t^L(R)\| < \frac{\varepsilon}{4}$, $|\psi(QR) - \psi_L(QR)| < \frac{\varepsilon}{4}$ and $|\psi(Q\alpha_t^L(R)) - \psi_L(Q\alpha_t^L(R))| < \frac{\varepsilon}{4}$ and a $\delta > 0$ such that $|\psi_L(Q\alpha_t^L(R) - QR)| < \frac{\varepsilon}{4}$ for $|t| < \delta$. Then, for $|t| < \delta$

$$\begin{aligned} |\psi(Q\alpha_t(R) - QR)| &\leq |\psi(Q\alpha_t(R) - Q\alpha_t^L(R))| + |\psi(Q\alpha_t^L(R)) - \psi_L(Q\alpha_t^L(R))| \\ &\quad + |\psi_L(Q\alpha_t^L(R)) - \psi_L(QR)| + |\psi_L(QR) - \psi(QR)| < \varepsilon. \end{aligned}$$

Next, we show that ψ is an (α, β) -KMS state as Definition 5.1. Note that the following inequality are valid for any $Q, R \in \mathfrak{R}$:

$$\begin{aligned} |\psi(Q\alpha_t(R)) - \psi_L(Q\alpha_t^L(R))| &\leq |\psi(Q\alpha_t(R)) - \psi_L(Q\alpha_t(R))| \\ &\quad + |\psi_L(Q\alpha_t(R)) - \psi_L(Q\alpha_t^L(R))|. \end{aligned} \quad (5.21)$$

By Theorem 3.3 and (5.21) and using the integral representation of an analytic function in a strip $I_\beta = \{z \in \mathbf{C} \mid 0 < \text{Im} z < \beta\}$ (see also the proof of [23, Theorem 2.2]), ψ is an (α, β) -KMS state.

Finally, we show the uniqueness of (α, β) -KMS state. Let ϕ be an arbitrary extremal (α, β) -KMS regular state at β . Let $(\mathfrak{H}_\psi, \pi_\psi, \Omega_\psi)$ and $(\mathfrak{H}_\phi, \pi_\phi, \Omega_\phi)$ be the GNS representation associated with ψ and ϕ . By $\widehat{\psi}$ and $\widehat{\phi}$, we denote the normal extension to the von Neumann algebra $\pi_\psi(\mathfrak{R})''$ and $\pi_\phi(\mathfrak{R})''$.

Let $\widehat{\phi}_N = \widehat{\phi}^{\beta W(N)}$, $N \in \mathbf{N}$, be the perturbed state of $\widehat{\phi}$ by $\beta W(N)$, where $W(N)$ is defined in Proposition 5.9. Put $\mathfrak{M} = \pi_\phi(\mathfrak{R})''$ and $\mathfrak{R}_L = \pi_\phi(\mathfrak{R}_L)''$, $L \in \mathbf{N}$. By Lemma 4.8, for $L \leq N$ we obtain

$$\begin{aligned} 0 &\leq S(\widehat{\phi}_N \upharpoonright_{\mathfrak{R}_L}, \widehat{\phi} \upharpoonright_{\mathfrak{R}_L}) \leq S(\widehat{\phi}_N, \widehat{\phi}) = \widehat{\phi}_N(\beta W(N)) - \log \widehat{\phi}_N(\mathbb{1}) \\ &\leq \widehat{\phi}_N(\beta W(N)) - \widehat{\phi}(\beta W(N)) \leq 4|\beta| \|\varphi\|_\infty. \end{aligned}$$

This follows from Pierls-Bogoliubov inequality:

$$\log \widehat{\phi}_N(\mathbb{1}) \geq \log e^{\widehat{\phi}(\beta W(N))} = \widehat{\phi}(\beta W(N)).$$

By Lemma 4.6, Lemma 4.7 and Lemma 5.9, for $L \leq N$

$$\begin{aligned} S(\widehat{\phi}_N \upharpoonright_{\mathfrak{R}_L}, \widehat{\phi} \upharpoonright_{\mathfrak{R}_L}) &= S(\phi_N \upharpoonright_{\mathfrak{R}_L}, \phi \upharpoonright_{\mathfrak{R}_L}) = S(\psi_N \upharpoonright_{\mathfrak{R}_L}, \phi \upharpoonright_{\mathfrak{R}_L}) \\ &= S(\widehat{\psi}_N \upharpoonright_{\pi_0(\mathfrak{R}_L)''}, \widehat{\phi} \upharpoonright_{\pi_0(\mathfrak{R}_L)'}) \leq 4|\beta| \|\varphi\|_\infty. \end{aligned}$$

Note that $\widehat{\psi}_N \upharpoonright_{\mathfrak{R}_L}$ converge to $\widehat{\psi} \upharpoonright_{\mathfrak{R}_L}$ in $\sigma(\mathcal{B}(\mathfrak{H}_{\Lambda_L}), \mathcal{B}(\mathfrak{H}_{\Lambda_L}))$ topology. By Lemma 4.10, it follows that

$$\begin{aligned} S(\psi \upharpoonright_{\mathfrak{R}_L}, \phi \upharpoonright_{\mathfrak{R}_L}) &= S(\widehat{\psi} \upharpoonright_{\pi_0(\mathfrak{R}_L)''}, \widehat{\phi} \upharpoonright_{\pi_0(\mathfrak{R}_L)'}) \\ &\leq \liminf_N S(\widehat{\psi}_N \upharpoonright_{\pi_0(\mathfrak{R}_L)''}, \widehat{\phi} \upharpoonright_{\pi_0(\mathfrak{R}_L)'}) \leq 4|\beta| \|\varphi\|_\infty. \end{aligned}$$

By Lemma 4.5 and [2, Lemma 3], we are done. ■ **Acknowledgment** We would

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